ON SUMS OF LOGNORMAL RANDOM VARIABLES*

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ABSTRACT

Approximations to the characteristic function of the lognormal distribution are computed and used to calculate approximations to the density of sums of lognormal random variables.

* This research was supported by NSF grant no. SIA 74-22773.

The authors thank their colleague Hung Cheng for a very fruitful discussion.

On Sums of Lognormal Random Variables*

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1. Introduction

The lognormal distribution has been used as a model for empirical data generating processes in a wide variety of disciplines. Aitcheson and Brown [1] cite over 100 applications. In portfolio analysis (Lintner [3]) and in statistical studies of the deposition of mineral resources (Barouch and Kaufman [2] and Uhler [4]) sums $X_1+\ldots+X_N = K$ of lognormal random variables (rvs) are of central interest.

Here we compute approximations to the density of a sum of N mutually independent and identically distributed lognormal rvs. As is well known, the density of a sum of independent, identically distributed rvs is given by the inverse Fourier (or LaPlace) transform of the Nth power of the characteristic function, so we begin by studying the characteristic function of a lognormal density. We perform an asymptotic analysis of it in its various regions for N and $\mathrm{Var}(X_i) = \sigma^2$ large and compute approximations to the density of the sum by transforming back.

We find that (a) for values of K larger than its mean, the density of K is approximately a three parameter lognormal density (cf. (44)); (b) for values of K near its mean, the density contains both lognormal-like and normal-like components, (cf. (45)); (c) for values of K larger than order one but smaller than its mean, the density is approximately a three parameter lognormal density, (cf. (46)) and (d) for values of K smaller than order one, the density is approximately lognormal (cf. (47)).

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2. Lognormal Characteristic Function

2.1 Properties of the Characteristic Function and Approximations

The characteristic function G(y) of the lognormal distribution is defined by

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} \exp\{-iyx - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x} \tag{1}$$

where y is complex with Imy ≤ 0 and μ has been set equal to 0 without loss of generality. The function G(y) is analytic everywhere in the lower half of the complex y plane, and continuous from below for real y. However, G(y) is not analytic near y = 0, and this greatly enhances the difficulty of approximating G(y).

An obvious way to attempt computation of G(y) is to expand e^{-iyx} in a Taylor series around zero and integrate term by term; i.e. to express G(y) in a moment series. In so doing, we are expanding G(y) around a point at which G(y) is non-analytic. Hence it is not at all surprising that this expansion fails in every respect. The first few terms of such an expansion cannot be looked upon as a "small y" approximation to G(y) as the resulting polynomial is analytic while G(y) is not. Furthermore, since the (n+1)st term in the series is $(-i)^n y^n \frac{1}{n!} e^{\sigma^2 n^2/2}$, this moment series is divergent for all $y \neq 0$. Trying hard, the moment series can be viewed as an asymptotic series provided that σ^2 is very small. This fact is not very useful, since for σ^2 small enough, the series is well approximated by e^{-iy}, the characteristic function of a density concentrated on the point 1.

Consider a point $y = \xi - i\eta$, $\eta > 0$ and rewrite $G(y)$ as

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\{-\eta x - i\xi x - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x} \qquad (2)$$

As long as η remains positive we may expand $e^{-i\xi x}$ in a power series, since we are expanding $G(y)$ around a point in its domain of analyticity. Hence we may write

$$G(y) = \sum_{j=0}^{\infty} \frac{(-i\xi)^j}{j!} \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\{-\eta x - \frac{1}{2\sigma^2}\log^2 x\} x^{j-1} dx \qquad (3)$$

The limit $\eta \to 0$ is not allowed after expanding since these two operations do not commute and a study of $G(y)$ based on (3) must be done with extreme caution. Each term in (3) can be derived from the first term, by differentiating with respect to η. This is allowed since the series is uniformly convergent.

Consequently, to construct a good approximation to $G(y)$ we need only study its behavior for y lying on the negative imaginary axis. This of course is not surprising, since any operation on $G(y)$ can be performed by deforming the contour of integration through the axis $\mathrm{Im} y < 0$.

We now focus our attention on

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\{-yx - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x}$$

for real positive y (namely $y = -i\eta$, $\eta > 0$, and so henceforth $\eta \equiv y$). We further assume that σ^2 is large.

Immediate properties of $G(y)$ are:

(i) $G(0) = 1$

(ii) $\lim_{y\to\infty} G(y) = 0$

(iii) $|G(y)| \leq 1$

(iv) $\frac{\partial G}{\partial y} = -e^{\sigma^2/2} G(ye^{\sigma^2})$

(v) $\frac{\partial^2 G}{\partial y^2} = e^{2\sigma^2} G(ye^{2\sigma^2})$

From properties (iv) and (v), it is apparent that differentiating $G(y)$ rescales y by e^{σ^2}. If e^{σ^2} is large, even if y is small, sufficient differentiation moves $G(y)$ to its asymptotic region. For example if $y = O(1)$, $\frac{\partial G}{\partial y}$ is a constant times $G(ye^{\sigma^2})$. Thus, when we wish to perform an integration involving $G(y)$, asymptotic evaluation of $G(y)$ may be necessary.

Let $y \to \infty$, and $\log y/\sigma^2$ be large. Change variables in $G(y)$: $yx = z$ to obtain

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \int_0^\infty \exp\{-z - \frac{1}{2\sigma^2}\log^2 z\} z^{\sigma^{-2}\log y} \frac{dz}{z} \tag{4}$$

Since $\frac{\log y}{\sigma^2}$ is assumed large and positive, the major contribution to (4) cannot come from $z \sim 0$. Thus, $\frac{1}{2\sigma^2} \log^2 z$ is small compared with z, and can be dropped. This is of course a crude approximation. To improve it we expand $\exp\{-\frac{1}{2\sigma^2} \log^2 z\}$ in a power series. As long as $\log y/\sigma^2 > 0$ the resulting term by term integration is uniformly convergent. Equation (4) takes the form

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\log^2 y} \sum_{j=0}^{\infty} \left(-\frac{1}{2\sigma^2}\right)^j \frac{1}{j!} \int_0^{\infty} e^{-z} (\log z)^{2j} z^{\sigma^{-2}\log y} \frac{dz}{z} \tag{5}$$

The integrals in (5) can be expressed in terms of derivatives of the Γ function:

$$\int_0^{\infty} e^{-z} (\log z)^{2j} \quad z^{\sigma^{-2}\log y} \frac{dz}{z} \quad = \Gamma^{(2j)}(\sigma^{-2}\log y) \tag{6}$$

with $\dfrac{\log y}{\sigma^2} > 0$.

Then $G(y)$ takes the form

$$G(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=0}^{\infty} \left(-\frac{1}{2\sigma^2}\right)^j \frac{1}{j!}\Gamma^{(2j)}(\sigma^{-2}\log y) \tag{7}$$

This expansion is exact but not really useful for computation since higher derivatives of the Γ function are very complicated. We now make use of the assumption that $\log y/\sigma^2$ is large, and approximate

$$\Gamma^{(2j)}(\sigma^{-2}\log y) \simeq \psi^{2j}(\sigma^{-2}\log y)\Gamma(\sigma^{-2}\log y) \tag{8}$$

where ψ is the logarithmic derivative of the Γ function. To justify this approximation we write a table:

$$\Gamma(\alpha)$$

$$\Gamma'(\alpha) = \psi(\alpha)\ \Gamma(\alpha)$$

$$\Gamma''(\alpha) = \{\psi^2(\alpha) + \psi'(\alpha)\}\Gamma(\alpha)$$

$$\Gamma'''(\alpha) = \{\psi^3(\alpha) + 3\psi(\alpha)\psi'(\alpha) + \psi''(\alpha)\}\Gamma(\alpha)$$

$$\Gamma''''(\alpha) = \{\psi^4(\alpha) + 6\psi^2(\alpha)\psi'(\alpha) + 4\psi(\alpha)\psi''(\alpha) + 3\psi'^2(\alpha) + \psi'''(\alpha)\}\Gamma(\alpha)$$

with $\alpha = \log y/\sigma^2$. If we substitute the leading term in the asymptotic series of $\psi(\alpha)$ which is $\log \alpha$, $\psi'(\alpha) \sim \frac{1}{\alpha}$, we can drop all derivatives of ψ compared with ψ, and $\Gamma^{(2j)}(\alpha) \doteq [\psi(\alpha)]^{2j}\Gamma(\alpha)$. Substitution of (8) into (7) yields a summable series for $G(y)$:

$$G(y) \doteq \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\}\Gamma(\sigma^{-2}\log y)\exp\{-\frac{1}{2\sigma^2}\psi^2(\sigma^{-2}\log y)\} \tag{9}$$

We can compute corrections to any order desired. For instance, in order to obtain the next term we write

$$\Gamma^{(2j)}(\alpha) = \psi^{2j}(\alpha)\ \Gamma(\alpha) + j(2j-1)\psi'(\alpha)\psi^{2j-2}(\alpha)\Gamma(\alpha) \tag{10}$$

which yields

$$G(y) \doteq \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\Gamma(\sigma^{-2}\log y)\exp\{-\frac{1}{2\sigma^2}\psi^2(\sigma^{-2}\log y)\}$$

$$* \{1 - \frac{1}{2\sigma^2}\psi'(\frac{\log y}{\sigma^2})[1 - \frac{1}{\sigma^2}\psi^2(\frac{\log y}{\sigma^2})] + O([\psi'(\frac{\log y}{\sigma^2})]^2\} \tag{11}$$

This expansion breaks down when $\psi'(\alpha)$ can no longer be neglected relative to $\psi(\alpha)$ and is particularly bad for $\alpha \downarrow 0$ $(y \sim 1)$. Also, (9) is valid for $y \gtrsim e^{\sigma^2/2}$; surprisingly, (9) works well for y close to $\exp\{\frac{1}{2}\sigma^2\}$.

We have already argued why $y \to 0$ is not legitimate here. Properties (ii) and (iii) are immediate for (9), and it satisfies (iv) to the order of approximation computed. [If (iv) is obeyed, the rest of the derivatives are immediate]. To see this, we neglect ψ', differentiate (9), and show

that the result equals $-\exp\{\frac{1}{2}\sigma^2\}$ times (9) with argument ye^{σ^2} in place of y. Property (iv) takes the form

$$\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\Gamma(\sigma^{-2}\log y)\exp\{-\frac{1}{2\sigma^2}\psi^2(\sigma^{-2}\log y)\}$$

$$\times\ (y\sigma^2)^{-1}\log y + (y\sigma^2)^{-1}\psi(\sigma^{-2}\log y)$$

$$- (y\sigma^4)^{-1}\psi(\sigma^{-2}\log y)\psi'(\sigma^{-2}\log y)\} \qquad (12)$$

$$= -\frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\ \frac{\log y}{y\sigma^2}\ \Gamma(\sigma^{-2}\log y)\exp\{-\frac{1}{2\sigma^2}[\psi(\sigma^{-2}\log y) + \frac{\sigma^2}{\log y}]^2\}$$

Approximating

$$\exp\{-\frac{1}{2\sigma^2}[\psi(\sigma^{-2}\log y) + \frac{\sigma^2}{\log y}]^2\}$$

$$\simeq \exp\{-\frac{1}{2\sigma^2}\psi^2(\sigma^{-2}\log y)\}[1 - \frac{\psi(\sigma^{-2}\log y)}{\log y}]$$

and neglecting ψ' in the LHS of (12) we obtain (after some simplification) the identity

$$\frac{\log y}{\sigma^2} - \frac{1}{\sigma^2}\ \psi(\sigma^{-2}\log y) = \frac{\log y}{\sigma^2}\ \{1 - \frac{1}{\log y}\psi(\frac{\log y}{\sigma^2})\} \qquad (14)$$

Thus, equation (iv) is obeyed by (9) when we keep all large and order 1 terms. The complicated way in which the equation is obeyed is due to the rich structure of G(y), despite its "simple" integral representation

Next we study $G(y)$ for small y. It seems reasonable to expect that for very small y, $G(y)$ can be approximated by a polynomial composed of the first few terms of its moment expansion. However, it must be done carefully: since $G(y)$ is not analytic in a neighborhood of $y = 0$ a meaningful small y approximation must possess this feature. Furthermore, since the asymptotic expansion of $G(y)$ for large y does not exhibit an additive polynomial, one must show how the polynomial disappears as y increases.

We begin with $y = O(\exp\{-\frac{3}{2}\sigma^2\})$, since this is a relevant order of y for sampling without replacement and proportional to random size (cf. [2]) and then generalize. Adding and subtracting $1 - yx$ from $\exp\{-yx\}$ write $G(y)$ for $y = O(\exp\{-\frac{3}{2}\sigma^2\})$ as

$$
\begin{aligned}
G(y) &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\{-yx - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x} \\
&= 1 - y\exp\{\frac{1}{2}\sigma^2\} \qquad (15) \\
&\quad + \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\int_0^\infty (e^{-z}-1+z)\exp\{-\frac{1}{2\sigma^2}\log^2 z\}z^{\sigma^{-2}\log y}\frac{dz}{z}
\end{aligned}
$$

Treating the above integral in the same way as (4), we obtain

$$
\begin{aligned}
G(y) &= 1 - y\exp\{\frac{1}{2}\sigma^2\} + \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\sum_{j=0}^\infty \{(-\frac{1}{2\sigma^2})^j\frac{1}{j!} \qquad (16) \\
&\quad \times [\frac{\partial^{2j}}{\partial\alpha^{2j}}\int_0^\infty (e^{-z}-1+z)\frac{dz}{z}]\}
\end{aligned}
$$

for $\alpha = \sigma^{-2}\log y$

In (16), $-2 < \alpha < -1$, and so the integral exists. In fact, this integral is an integral representation of $\Gamma(\alpha)$ for α negative and non-integer valued. Given (16) the method used to compute an expansion for large y can be used and we find that for $y = O(\exp\{-(m+\frac{1}{2})\sigma^2\})$ and integer m, we can add and subtract a polynomial of the m^{th} degree from $\exp\{-yx\}$ to obtain

$$G(y) \doteq \sum_{j=0}^{m} \frac{(-y)^j}{j!}\exp\{\tfrac{1}{2}j^2\sigma^2\} + \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\}\Gamma(\frac{\log y}{\sigma^2})\exp\{-\frac{1}{2\sigma^2}\psi^2(\frac{\log y}{\sigma^2}) \tag{17}$$

for $-m > \dfrac{\log y}{\sigma^2} > -(m+1)$.

This expansion possesses both of the features needed for it to be a meaningful approximation to G(y) for small y; it is non-analytic at $y = 0$ and has a polynomial piece. However, it has two major defects: it is not defined at $y = \exp\{-m\sigma^2\}$ for integer m, and it appears as if G(y) is discontinuous at $y = \exp\{-m\sigma^2\}$. Hence formula (17) can be regarded at best as an approximation of limited validity. Furthermore, it does not explain the relation between the rising power of the polynomial term as y decreases and the lognormal term. Therefore further analysis is needed.

We split the integral representation of G(y) into a sum $I_1 + I_2$ of two integrals defined by

$$I_1(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{1/y} \exp\{-yx - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x} \quad \text{and}$$

$$I_2(y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{1/y}^{\infty} \exp\{-yx - \frac{1}{2\sigma^2}\log^2 x\}\frac{dx}{x} \tag{18}$$

When y is sufficiently small, I_2 is small compared with I_1 and in the limit $y \to 0$, $I_1 = G(0) = 1$. Consequently, we first analyze I_1 for y small. Expanding $\exp\{-yx\}$ and integrating term by term we obtain a uniformly convergent series:

$$I_1(y) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{(-y)^j}{j!} \int_0^{1/y} \exp\{-\frac{1}{2\sigma^2}\log^2 x\} x^j \frac{dx}{x} \tag{19}$$

$$= \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-y)^j}{j!} \exp\{\tfrac{1}{2}j^2\sigma^2\} \operatorname{erfc}[\frac{\log(ye^{j\sigma^2})}{\sqrt{2}\sigma}]$$

with

$$\operatorname{erfc}(z) \equiv \frac{2}{\sqrt{2\pi}} \int_z^{\infty} \exp\{-t^2\}dt \equiv 1 - \operatorname{erf}(z) \tag{20}$$

for $z \geq 0$. In (19) we must distinguish between three types of terms: setting $y = \exp\{-\lambda\sigma^2\}$, with $\lambda > 0$

$$\sigma(j - \lambda) < 0, \tag{21a}$$

$$\sigma(j - \lambda) > 0, \tag{21b}$$

$$\sigma(j - \lambda) \doteq 0. \tag{21c}$$

The argument in (19) is $\frac{1}{\sqrt{2}}\sigma(j - \lambda)$, hence there are a finite number of terms for which $j - \lambda < 0$. Since σ^2 is assumed large, there may be one term for which $\sigma(j-\lambda) \doteq 0$ (when $j - \lambda = 0(\sigma^{-1})$, there is only one such term). Remaining terms are of type $\sigma(j-\lambda) > 0$.

We replace erfc(·) in each term for which $j - \lambda > 0$ by its asymptotic expansion, namely,

$$\operatorname{erfc}(z) = \frac{1}{z\sqrt{\pi}} \exp\{-z^2\}[1 + \sum_{\ell=1}^{\infty} (-1)^{\ell} \frac{(2\ell-1)!!}{(2z^2)^{\ell}}] \tag{22}$$

Since $\sum_{j=0}^{\infty} \frac{(-1)^j}{j!}[j + \sigma^{-2}\log y]^{-1}$ is a series representation of the incomplete gamma function $\gamma(\sigma^{-2}\log y;\ 1)$, (25) may be rewritten as

$$I_1(y) \simeq \sum_{j=0}^{k} \frac{(-y)^j}{j!} \exp\{\tfrac{1}{2}j^2\sigma^2\} + \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\gamma(\sigma^{-2}\log y;\ 1) \tag{26}$$

$$+ \frac{(-1)^{k+1}}{(k+1)!} \exp\{-\frac{1}{2\sigma^2}\log^2 y\}[\tfrac{1}{2}\exp\{\frac{1}{2\sigma^2}\log^2(ye^{(k+1)\sigma^2})\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log ye^{(k+1)\sigma^2})$$

$$- (\sigma\sqrt{2\pi}(1 + k + \sigma^{-2}\log y))^{-1}]$$

Even though $\gamma(\sigma^{-2}\log y;\ 1)$ has a pole for $\sigma^{-2}\log y$ a negative integer, when $y = \exp\{-(k+1)\sigma^2\}$ the pole of $\gamma(\sigma^{-2}\log y;\ 1)$ is cancelled by that of $(1 + k + \sigma^{-2}\log y)^{-1}$ so that (26) is a legitimate representation of $I_1(y)$ for <u>all</u> small y.

We now turn to $y \simeq 1$ and first consider $y > 1$. Then $\log(ye^{j\sigma^2}) > 0$ or $j - \lambda > 0$ for all j except $j = 0$, so

$$I_1(y) = \frac{1}{2} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \exp\{\frac{1}{2\sigma^2}\log^2(ye^{j\sigma^2})\}\mathrm{erfc}(\frac{\log(ye^{j\sigma^2})}{\sqrt{2}\sigma})$$

$$+ \tfrac{1}{2}\mathrm{erfc}(\frac{\log y}{\sqrt{2}\sigma}) \tag{27}$$

The approximation (26) to $I_1(y)$ for small y was computed keeping only the first term of the asymptotic series for erfc(·); it is not clear a priori that this leads to an accurate approximation to $I_1(y)$ when $y = O(1)$, so we replace the asymptotic series for $\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log y)$ with its series expansion for small argument $z \geq 0$,

$$\mathrm{erf}(z) = e^{-z^2} \sum_{j=0}^{\infty} \frac{z^{2j+1}}{\Gamma(\frac{3}{2}+j)}, \quad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z),$$

and approximate $I_1(y)$ with

$$I_1(y) \simeq \frac{1}{2\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} [j + \sigma^{-2}\log y]^{-1}$$

$$+ \frac{1}{2}[1 - \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=0}^{\infty} \frac{1}{\Gamma(\frac{3}{2}+j)} (\frac{\log y}{\sqrt{2}\sigma})^{2j+1}] \qquad (28)$$

$$- \frac{1}{2}y \exp\{\frac{1}{2}\sigma^2\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log ye^{\sigma^2})$$

When σ^2 is large the last term in (28) can be incorporated into the first sum.

For $y \simeq 1$, $y < 1$, and σ^2 large enough so that $\log(ye^{\sigma^2}) > 0$,

$$I_1(y) = \frac{1}{2}\exp\{-\frac{1}{2\sigma^2}\log^2 y \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \exp\{\frac{1}{2\sigma^2}\log^2(ye^{j\sigma^2})\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log(ye^{j\sigma^2}))$$

$$+ \frac{1}{2}[2 - \mathrm{erfc}(-\frac{1}{\sqrt{2}\sigma}\log y)]$$

$$\simeq \frac{1}{2\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} [j + \sigma^{-2}\log y]^{-1} \qquad (29)$$

$$+ \frac{1}{2}[1 - \exp\{-\frac{1}{2\sigma^2}\log^2 y \sum_{j=0}^{\infty} \frac{1}{\Gamma(\frac{3}{2}+j)} (-\frac{\log y}{\sqrt{2}\sigma})^{2j+1}]$$

$$- \frac{1}{2}y \exp\{\frac{1}{2}\sigma^2\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log ye^{\sigma^2})$$

As $y \to 1$, both (28) and (29) approach the same limit, namely,

$$\begin{aligned}\lim_{y\to 1} I_1(y) &\simeq \frac{1}{2}\{1 - \exp\{\tfrac{1}{2}\sigma^2\}\operatorname{erfc}(\frac{\sigma}{\sqrt{2}}) + \frac{1}{\sigma\sqrt{2\pi}}\sum_{j=2}^{\infty}\frac{(-1)^j}{j!j}\} \\ &= \frac{1}{2}\{1 - \exp\{\tfrac{1}{2}\sigma^2\}\operatorname{erfc}(\frac{\sigma}{\sqrt{2}}) + \frac{1}{\sigma\sqrt{2\pi}}[1 + \operatorname{Ei}(-1) - \gamma]\}\end{aligned} \tag{30}$$

where $\gamma = .57721\ 56649$, Euler's constant and

$$\operatorname{Ei}(-1) = -\int_1^{\infty} e^{-t}t^{-1}dt \simeq .21938\ 3934.$$

We conclude our discussion of $G(y)$ with an asymptotic analysis of $I_2(y)$ as defined by (18). When y is large the principal contribution to $G(y)$ is from $I_2(y)$ and is of the form (11). When y is small, $I_2(y)$ is small, but when $y \simeq 1$, $I_2(y)$ is of the same order as $I_1(y)$.

Consider $y \simeq 1$ and σ^2 large first. Rewrite

$$I_2(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\int_0^1 \exp\{-\frac{1}{z} - \frac{1}{2\sigma^2}\log^2 z\}z^{-\sigma^{-2}\log y}\frac{dz}{z} \tag{31}$$

When $y \simeq 1$ and σ^2 is large the major contribution to $I_2(y)$ comes from $z \simeq 1$ and so we approximate

$$\frac{1}{z} \simeq \frac{1}{2-z} \simeq 1 - (1-z) + (1-z)^2 \simeq z. \tag{32}$$

If we replace $1/z$ with z in (31) when $y \simeq 1$ and σ^2 is large, we see that $I_2(y) \simeq I_1(y^{-1})$ and $I \simeq I_1(y) + I_1(y^{-1})$ where I_1 is given by (28) and (29).

For $y \ll 1$ we write

$$I_2(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\int_1^{\infty}\exp\{-z - \frac{1}{2\sigma^2}\log^2 z\}z^{\sigma^{-2}\log y}\frac{dz}{z} \tag{33}$$

When σ^2 is large $\sigma^{-2}\log^2 z \ll z$ for $1 \le z \le \exp\{\sigma^2\}$ and in this region we may expand $\exp\{-\frac{1}{2\sigma^2}\log^2 z\}$; the contribution from the region $z > \exp\{\sigma^2\}$ is small for large σ^2 and so we ignore it. Explicitly

$$I_2(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=0}^{\infty} \frac{1}{j!}(-\frac{1}{2\sigma^2})^j \frac{\partial^{2j}}{\partial\alpha^{2j}} \Gamma(\alpha;1) \tag{34}$$

where for negative α, $\Gamma(\alpha;1)$ is the incomplete gamma function

$$\Gamma(\alpha;1) = \frac{1}{e\Gamma(1-\alpha)} \int_0^{\infty} e^{-t} t^{-\alpha} \frac{dt}{1+t} \tag{35}$$

For α large and negative a standard steepest descent calculation yields

$$\Gamma(\alpha;1) \simeq \frac{\sqrt{\pi}}{e\Gamma(1-\alpha)} e^{-\theta}\theta^{-\alpha}[2 + \theta + \frac{2}{\theta}]^{-1/2} \tag{36}$$

with $\theta = -(1 + \alpha + \frac{1}{\alpha})$ so that the leading term in an asymptotic expansion of $I_2(y)$, $y \ll 1$ is

$$\begin{aligned} I_2(y) &\simeq \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\}[\Gamma(1 - \sigma^{-2}\log y)]^{-1} \\ &\times \exp\{-\frac{\log y}{\sigma^2} - \frac{\sigma^2}{\log y}\}[-\frac{\log y}{\sigma^2} - \frac{\sigma^2}{\log y} - 1]^{1/2 - \sigma^{-2}\log y} \\ &\times [(\frac{\log y}{\sigma^2} + \frac{\sigma^2}{\log y})^2 + 1]^{-1/2} . \end{aligned} \tag{37}$$

By differentiating (37) with respect to α, higher order terms may be computed if desired.

2.2 Summary of Approximations to G(y)

The approximations to G(y) for σ^2 large computed in 2.1 are:

(1) For $y << 1$ and $-(k+1) \leq \frac{\log y}{\sigma^2} \leq -k$,

$$G(y) \doteq \sum_{j=0}^{k} \frac{(-y)^j}{j!} \exp\{\tfrac{1}{2}j^2\sigma^2\} + \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\}\gamma(\sigma^{-2}\log y;1)$$

$$+ \frac{(-1)^k}{k!}\ [\tfrac{1}{2}\exp\{\frac{1}{2\sigma^2}\log^2(ye^{k\sigma^2})\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log(ye^{k\sigma^2})) - \frac{1}{\sigma\sqrt{2\pi}[k+\sigma^{-2}\log y]}]$$

$$+ \frac{(-1)^{k+1}}{(k+1)!}[\tfrac{1}{2}\exp\{\frac{1}{2\sigma^2}\log^2(ye^{(k+1)\sigma^2})\}\mathrm{erfc}(\frac{1}{\sqrt{2}\sigma}\log(ye^{(k+1)\sigma^2})$$

$$- \frac{1}{\sigma\sqrt{2\pi}[k+1+\sigma^{-2}\log y]}] + I_2(y) \qquad (38)$$

where $\gamma(\sigma^{-2}\log y;1)$ is an incomplete gamma function whose singularities at integer $-k$ and $-(k+1)$ are cancelled by the singularities of $[k + \sigma^{-2}\log y]^{-1}$ and $[k + 1 + \sigma^{-2}\log y]^{-1}$ respectively, and $I_2(y)$ is given by (37). An asymptotic evaluation of $I_2(y)$ when y is small (cf. formula [37]) shows it to be small by comparison with $I_1(y)$.

(2) For $y \doteq 1$ and $y > 1$, with $I_1(y)$ given by (28),

$$G(y) \doteq I_1(y) + I_1(y^{-1}) \doteq \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=2}^{\infty} \frac{(-1)^j}{(j-1)!}[j^2-\sigma^{-2}\log^2 y]^{-1}$$

$$- \tfrac{1}{2}\exp\{\tfrac{1}{2}\sigma^2\}[y\ \mathrm{erfc}(\frac{1}{\sigma\sqrt{2}}\log ye^{\sigma^2}) + \frac{1}{y}\mathrm{erfc}(\frac{1}{\sigma\sqrt{2}}\log y^{-1}e^{\sigma^2})] \qquad (39)$$

$$+ 1 - \exp\{-\frac{1}{2\sigma^2}\log^2 y\} \sum_{j=0}^{\infty} \frac{1}{\Gamma(\frac{3}{2}+j)}(\frac{\log y}{\sigma\sqrt{2}})^{2j+1}$$

(3) For $y = 1$,

$$G(1) \doteq 1 - \exp\{\tfrac{1}{2}\sigma^2\}\mathrm{erfc}(\frac{\sigma}{\sqrt{2}}) + \frac{1}{\sigma\sqrt{2\pi}}[1 + \mathrm{Ei}(-1) - \gamma] \qquad (40)$$

where $\gamma = .5772156649$ and $\mathrm{Ei}(-1) = .219383934$.

(4) For $y \doteq 1$ and $y < 1$, (39) with y replaced by y^{-1} and $I_1(y)$ given by (29).

(5) For $y >> 1$,

$$G(y) \doteq \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}\log^2 y\}\Gamma(\sigma^{-2}\log y)\exp\{-\frac{1}{2\sigma^2}\psi^2(\sigma^{-2}\log y)\}$$
$$\times\ [1 - \frac{1}{2\sigma^2}\psi'(\sigma^{-2}\log y)(1 - \sigma^{-2}\psi^2(\sigma^{-2}\log y))] \tag{41}$$

Figure 1 displays the graph of $G(y)$ for $\sigma^2 = 3.0$ and $\mu = 0$ and an approximation to it using the above approximation formulae. The solid line is drawn through points computed to five digits accuarcy by numerical integration of (1).

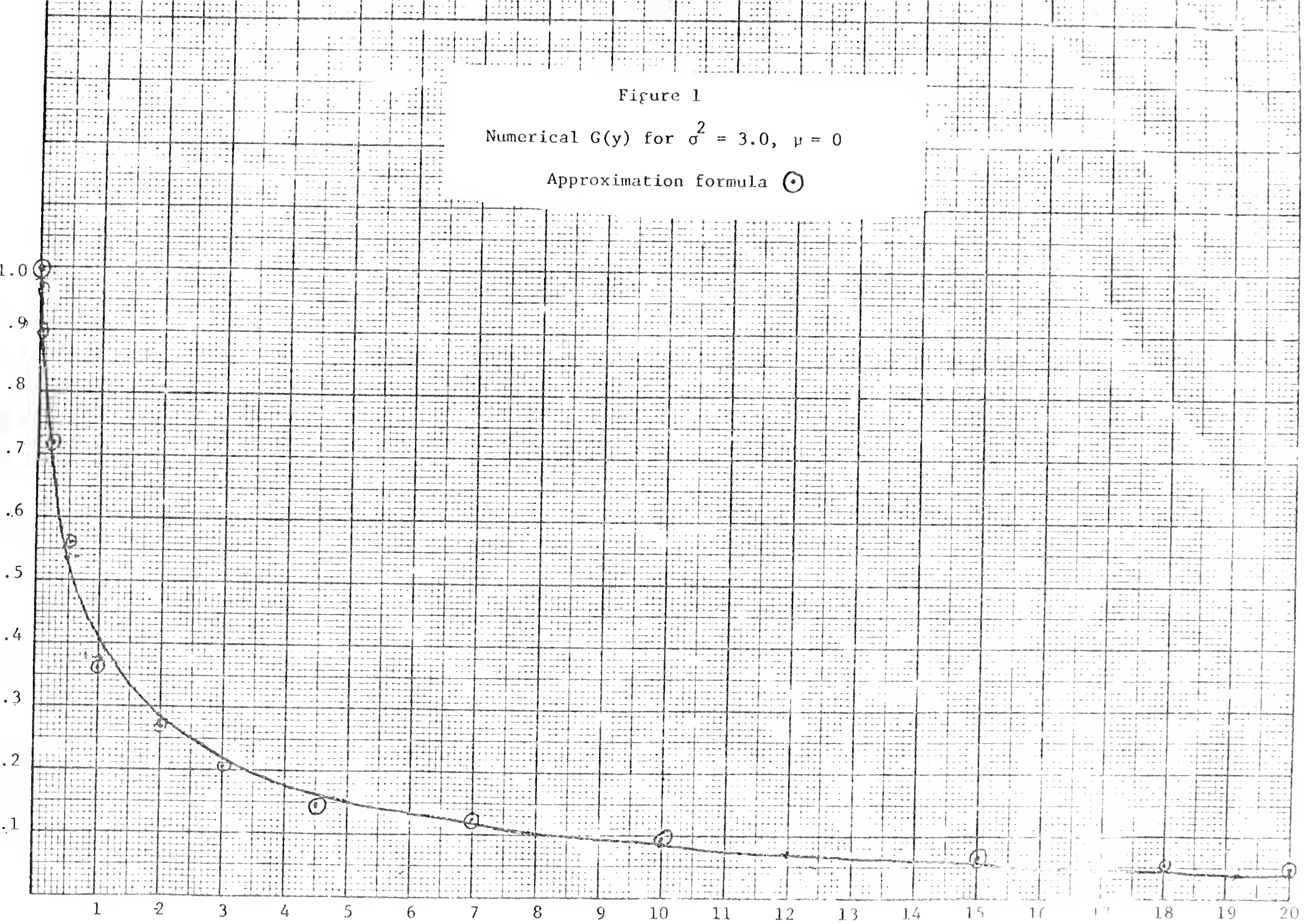

Figure 1

Numerical G(y) for $\sigma^2 = 3.0$, $\mu = 0$

Approximation formula ⊙

3. Inversion of $[G(y)]^N$

The density h(K) of the sum of N lognormal random variables is

$$h(K) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \exp\{Ky\}[G(y)]^N dy. \tag{42}$$

The major contribution to this integral comes from the region $Ky = O(1)$. Since G(y) has a different form for each of the regions $y \ll 1$, $y \simeq 1$, and $y \gg 1$, the functional form of h(K) is different for differing orders of K. For each of the following cases N and σ^2 are assumed large.

(1) $K > NM_1$. For this case we approximate G(y) by (38) and write

$$G(y) \simeq e^{-yM_1} a(y) + \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} b(y) \tag{43}$$

with $a(0) = 1$. The number of terms in a(y) is determined by the magnitude of K. Approximating $[G(y)]^N$ by the first two terms of the binomial expansion of G(y) expressed as (43),

$$[G(y)]^N \simeq \exp\{-NM_1 y\}[a^N(y) + \frac{Ne^{M_1 y}}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2\sigma^2}\log^2 y\} b(y) a^{N-1}(y)],$$

the integral (42) is approximated by a sum of two integrals. If $a(y) \equiv 1$ the first, that with integrand $\exp\{(K - NM_1)y\}a^N(y)$, vanishes; when

$a(y) \doteq 1$ for $y \doteq 0$ it is exponentially small. Consequently we approximate

$$h(K) \doteq \frac{1}{2\pi i} \cdot \frac{N}{\sigma\sqrt{2\pi}} \int_{\lambda-i\infty}^{\lambda+i\infty} \exp\{(K - [N-1]M_1)y - \frac{1}{2\sigma^2}\log^2 y\} b(y) a^{N-1}(y)\,dy$$

$$\doteq \frac{N}{\sigma\sqrt{2\pi}} \frac{\exp\{-\frac{1}{2\sigma^2}\log^2(K - [N-1]M_1)\}}{K - [N-1]M_1} \tag{44}$$

$$\times\ b(\{K - [N-1]M_1\}^{-1}) a^{N-1}(\{K - [N-1]M_1\}^{-1})$$

Corrections to (44) can be computed in a straightforward fashion.

The approximation (44) to $h(K)$ shows that for large $K > NM_1$, $h(K)$ is lognormal-like; i.e. the leading term of an asymptotic expansion of it is composed of a three parameter $(\mu=0,\ \sigma^2,\ (N-1)M_1)$ lognormal density with argument $K - (N-1)M_1$, times a correction.

(2) $K \doteq NM_1$. This case requires specification of the scales of N and K - different scales give different answers. In particular, for $N = O(\exp\{2\sigma^2\})$ and $K = O(\exp\{\frac{5}{2}\sigma^2\})$, $a(y)$ may be approximated by $\exp\{-yM_1 + \frac{1}{2}Vy^2\}$. Upon expanding $[G(y)]^N$ written as (43) and integrating term by term we obtain

$$f(K) \doteq (2\pi\sigma^2)^{-\frac{1}{2}N} b^N(K)\exp\{-\frac{N}{\sigma^2}\log^2 K\}$$

$$+ \sum_{j=0}^{N-1} [\binom{N}{j}(2\pi V(N-j))^{-\frac{1}{2}}\exp\{-\frac{(K - (N-j)M_1)^2}{2(N-j)V}\} \tag{45}$$

$$\times \{(2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\frac{1}{2\sigma^2}\log^2\frac{|K - (N-j)M_1|}{(N-j)V}\}\ b(\frac{(N-j)V}{|K - (N-j)M_1|})\}^j]$$

(3) $1 < K < (N-1)M_1$. This case may be treated in the same way as case (1) with $(N-1)M_1 - K$ replacing $K - (N-1)M_1$:

$$h(K) \simeq \frac{N}{\sigma\sqrt{2\pi}}[(N-1)M_1 - K]^{-1}\exp\{-\frac{1}{2\sigma^2}\log^2((N-1)M_1 - K)\}$$

$$b([(N-1)M_1 - K]^{-1}\ a^{N-1}([(N-1)M_1 - K]^{-1}) \tag{46}$$

(4) $K \ll 1$. Use the expansion (11) of $G(y)$ in its asymptotic region:

$$h(K) \simeq \frac{1}{2\pi i} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{\lambda-i\infty}^{\lambda+i\infty} \exp\{Ky - \frac{1}{2\sigma^2}\log^2 y - \frac{1}{2\sigma^2}\ \psi^2(\sigma^{-2}\log\ y)\Gamma(\sigma^{-2}\log\ y)dy$$

$$\simeq (\frac{N}{\sigma^2\sqrt{2\pi}})^{\frac{1}{2}}\exp\{-\frac{N}{2\sigma^2}\log^2 K\}\frac{1}{K} \tag{47}$$

$$\times\ N^{-\frac{1}{2}}(\sigma\sqrt{2\pi})^{-(N-1)}\Gamma^N(-\sigma^2\log\ K)\exp\{-\frac{N}{2\sigma^2}\ \psi^2(-\frac{\log\ K}{\sigma^2})\}$$

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Date Due